FILL-INS OF NONNEGATIVE SCALAR CURVATURE, STATIC METRICS, AND QUASI-LOCAL MASS

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ABSTRACT. Consider a triple of "Bartnik data" (Σ, γ, H), where Σ is a topological 2-sphere with Riemannian metric γ and positive function H. We view Bartnik data as a boundary condition for the problem of finding a compact Riemannian 3-manifold (Ω, g) of nonnegative scalar curvature whose boundary is isometric to (Σ, γ) with mean curvature H (c.f. [5]). Considering the perturbed data $(\Sigma, \gamma, \lambda H)$ for a positive real parameter λ, we find that such a "fill-in" (Ω, g) must exist for λ small and cannot exist for λ large; moreover, we prove there exists an intermediate threshold value.

The main application is the construction of a new quasi-local mass, a concept of interest in general relativity. This mass has the nonnegativity property, but differs from many other definitions in that it tends to vanish on static vacuum (as opposed to flat) regions. We also recognize this mass as a special case of a type of twisted product of quasi-local mass functionals. Several ideas in this paper draw on work of Bray [8], Brendle–Marques–Neves [9], Corvino [11], Miao [19], and Shi–Tam [23] .

1. INTRODUCTION

Riemannian 3-manifolds of nonnegative scalar curvature arise naturally in general relativity as totally geodesic spacelike submanifolds of spacetimes obeying Einstein's equation and the dominant energy condition. In this setting, scalar curvature plays the role of energy density. Black holes in this setting are manifested as connected minimal surfaces that minimize area to the outside. If S is a disjoint union of such surfaces of total area A, the number $\sqrt{\frac{A}{16\pi}}$ is interpreted to encode the total mass of the collection of black holes, possibly accounting for potential energy between them [8].

A fundamental question in general relativity is to quantify how much mass is contained in a compact region Ω in a spacelike slice of a spacetime [20]. Constructing examples of such *quasi-local mass* has led to a very active field of research (we mention here a small number of possible references: [16, 24, 25]). For most definitions, the quasi-local mass of Ω depends only boundary data of Ω: namely the induced 2-metric and induced mean curvature function. We reference pioneering work of Bartnik [4, 5], whose name is given in the following definition.

All metrics and functions in this paper are assumed to be smooth, unless otherwise stated.

Definition 1. A triple $\mathcal{B} = (\Sigma, \gamma, H)$, where Σ is a topological 2-sphere, γ is a Riemannian metric on Σ of positive Gauss curvature, and H is a positive function on Σ is called **Bartnik data**.

While not always necessary, it is often customary to restrict to positive Gauss curvature and positive functions H, as we do here. A typical problem involving Bartnik data (Σ, γ, H) is to construct a Riemannian 3-manifold (M, g) satisfying some nice geometric properties such that the boundary ∂M is isometric to (Σ, γ), and the mean curvature of ∂M agrees with H. For instance, one might require (M, g) to be asymptotically flat with nonnegative

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or zero scalar curvature (see [7], for instance). Such a manifold is called an *extension* of the Bartnik data.

We will instead focus on the dual problem of constructing compact *fill-ins* of the Bartnik data, realizing (Σ, γ, H) as the boundary of a compact 3-manifold. This problem was considered by Bray in the construction of the Bartnik inner mass [8] (see section 2.3 below).

Definition 2. A **fill-in** of Bartnik data (Σ, γ, H) is a compact Riemannian 3-manifold (Ω, g) with boundary such that there exists an isometric embedding $\iota : (\Sigma, \gamma) \to (\Omega, g)$ with the following properties:

  (1) the image $\iota(\Sigma)$ is some connected component S_0 of $\partial\Omega$, and
  (2) $H = H_{S_0} \circ \iota$ on Σ, where H_{S_0} is the mean curvature of S_0 in (Ω, g), in the outward direction.

Without loss of generality, if (Ω, g) is a fill-in of (Σ, γ, H), we shall henceforth identify Σ with $\iota(\Sigma)$ and H with the mean curvature of $\iota(\Sigma)$.

We will primarily be concerned with fill-ins satisfying the following geometric constraints.

Definition 3. A fill-in (Ω, g) of (Σ, γ, H) is **valid** if the metric g has nonnegative scalar curvature and either

  (1) $\partial\Omega = \Sigma$, or
  (2) $\partial\Omega \setminus \Sigma$ is a minimal (zero mean curvature) surface, possibly disconnected.

Figure 1 provides a graphical depiction. In physical terms, a valid fill-in is a compact region in a slice of a spacetime that has nonnegative energy density and possibly contains black holes. Another characterization of the second class of valid fill-ins is a cobordism of nonnegative scalar curvature that joins the given Bartnik data to a minimal surface. Note that we require $\partial\Omega \setminus \Sigma$ to be minimal, but not necessarily area-minimizing. Figure 1 provides a graphical depiction.

FIGURE 1. Valid fill-ins of Bartnik data
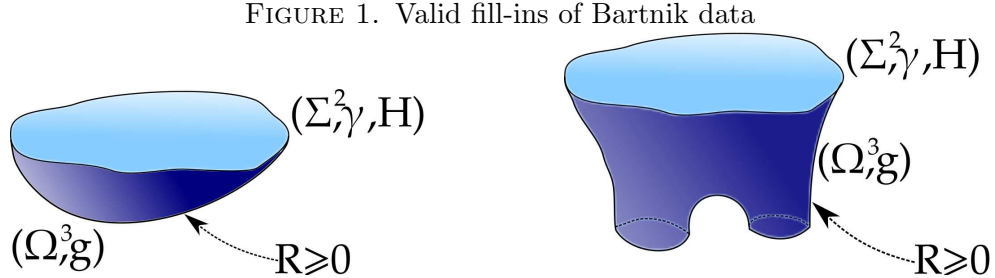
On the left is a valid fill-in of (Σ, γ, H) of the first type (i.e., $\partial\Omega = \Sigma$). On the right is a valid fill-in of the second type ($\partial\Omega \setminus \Sigma$ is minimal). R denotes the scalar curvature of g.

Interestingly, Bartnik data falls into exactly one of three types. Although trivial to prove, the following fact motivates much of the present paper.

Observation 4 (Trichotomy of Bartnik data)**.** *Bartnik data (Σ, γ, H) belongs to exactly one of the following three classes:*

  (1) Negative type: (Σ, γ, H) admits no valid fill-in.
  (2) Zero type: (Σ, γ, H) admits a valid fill-in, but no valid fill-in with minimal boundary.
  (3) Positive type: (Σ, γ, H) admits a valid fill-in with minimal boundary.

Outline. In section 2, we give some geometric characterizations of valid fill-ins of Bartnik data of zero and positive type, making connections with static vacuum metrics. We also recall in section 2.3 the Bartnik inner mass, which explains the use of the words positive, zero, and negative in the trichotomy.

The essential idea of this paper, presented in section 3, is to study the behavior of Bartnik data $(\Sigma, \gamma, \lambda H)$, where the real parameter $\lambda > 0$ is allowed to vary. We show in Theorem 10 that the data passes through all three classes of the trichotomy, with interesting behavior at some unique borderline value $\lambda = \lambda_0$. In section 3.1, we introduce a function that probes the geometry of valid fill-ins of $(\Sigma, \gamma, \lambda H)$.

The main application occurs in section 4, where we use the number λ_0 to define a quasi-local mass for regions in 3-manifolds of nonnegative scalar curvature. Several properties are shown to hold, including nonnegativity. What distinguishes this definition from most others is its tendency to vanish on static vacuum, as opposed to flat, data. We give a brief physical argument for why such a property may be desirable.

Section 5 consists of examples of Bartnik data of all three types, and compares our definition with the Hawking mass and Brown–York mass. In section 6 we introduce a general construction for "twisting" two quasi-local mass functionals together, of which the above quasi-local mass is a special case. The final section is a discussion of some potentially interesting open problems.

Acknowledgements. The author is indebted to Hubert Bray for suggesting the main idea of varying the parameter λ.

2. Fill-ins of nonnegative type and the inner mass

2.1. **Zero type data and static vacuum metrics.** First, we classify the geometry of valid fill-ins of Bartnik data of zero type. Recall that a Riemannian 3-manifold (Ω, g) is *static vacuum* if there exists a function $u \geq 0$ (called the *static potential*), with $u > 0$ on the interior of Ω, such that the Lorentzian metric

$$h = -u^2 dt^2 + g$$

on $\mathbb{R} \times \Omega$ has zero Ricci curvature. This condition is equivalent to the system of equations:

$$\Delta u = 0 \tag{1}$$

$$\mathrm{Ric} = \frac{\mathrm{Hess}\, u}{u} \tag{2}$$

where Δ, Ric and Hess are the Laplacian, Ricci curvature, and Hessian with respect to g. Equation (1) together with the trace of (2) shows that static vacuum metrics have zero scalar curvature.

The following result is primarily a consequence of Corvino's work on local scalar curvature deformation [11].

Proposition 5. *If $\mathcal{B}$ is Bartnik data of zero type, then any valid fill-in is static vacuum.*

The idea of the proof is to use a valid fill-in that is not static vacuum to construct a valid fill-in that contains a black hole. By a very rough analogy, one might think of this physically as taking some of the energy content in a fill-in and squeezing it down into a black hole. The delicate issue is that we must preserve the boundary data in the process.

Proof. Let (Ω, g) be a valid fill-in of zero type data (Σ, γ, H). By definition, $\partial\Omega$ consists of a single component, that corresponding to Σ. Suppose (Ω, g) is not static vacuum; Corvino proves the existence of a metric $\overline{g}$ on Ω with nonnegative, scalar curvature, positive at some interior point p, such that $g - \overline{g}$ is supported away from $\partial\Omega$ [11]. In particular, $(\Omega, \overline{g})$ is a valid fill-in for the data. Choose $r > 0$ sufficiently small such that on the closed metric ball $\overline{B}(p, r)$ (with respect to $\overline{g}$), the scalar curvature of $\overline{g}$ is bounded below by some $\epsilon > 0$.

On the set $\overline{B}(p, r/2) \setminus \{p\}$, let G be a Green's function for the Laplacian (with respect to $\overline{g}$) that blows up at p and vanishes on $\partial\overline{B}(p, r/2)$ (see Theorem 4.17 of [3].) By the maximum principle, G is positive, except on $\partial\overline{B}(p, r/2)$. Extend G by zero to the rest

of $\Omega \setminus \{p\}$, so that G Lipschitz, smooth away from $\partial \overline{B}(p, r/2)$. Perturb G to a smooth, nonnegative function $\tilde{G}$ on $\Omega \setminus \{p\}$ that agrees with G except possibly on the annular region $\overline{B}(p, 3r/4) \setminus \overline{B}(p, r/4)$. For a parameter $\delta > 0$ to be determined, define the conformal metric

$$\tilde{g} = (1 + \delta \tilde{G})^4 \overline{g}$$

on $\Omega \setminus \{p\}$. By construction, $\tilde{g} = \overline{g}$ outside $\overline{B}(p, r)$ and thus has nonnegative scalar curvature outside this ball. For points inside $\overline{B}(p, r)$, we apply the rule for the change in scalar curvature under a conformal deformation (see appendix A):

$$\tilde{R} = (1 + \delta \tilde{G})^{-5}(-8\overline{\Delta}(1 + \delta \tilde{G}) + (1 + \delta \tilde{G})\overline{R})$$
$$\geq (1 + \delta \tilde{G})^{-5}(-8\delta \overline{\Delta} \tilde{G} + \epsilon),$$

where ϵ is the lower bound on the scalar curvature of $\overline{g}$. Here, $\tilde{R}$ and $\overline{R}$ are the scalar curvatures of $\tilde{g}$ and $\overline{g}$, and $\overline{\Delta}$ is the Laplacian for $\overline{g}$. Since $\overline{\Delta} \tilde{G}$ has compact support, we may choose $\delta > 0$ sufficiently small so that the above is strictly positive. In particular, the metric $\tilde{g}$ has nonnegative scalar curvature on $\Omega \setminus \{p\}$.

Now, suppose s is the distance function with respect to $\overline{g}$ from the point p. For s sufficiently small, G is of the form $\frac{c}{s}$ to leading order for some constant $c > 0$. The normal derivative of G to the sphere of radius s about p in the outward direction is $-\frac{c}{s^2}$ to leading order. The mean curvature of the sphere of radius s with respect to $\overline{g}$ is $\frac{2}{s}$ to leading order, and so the mean curvature of this sphere with respect to $\tilde{g}$ is

$$(1 + \delta \tilde{G})^{-3} \left(\frac{2}{s} \cdot \frac{c}{s} - 4\frac{c}{s^2} \right)$$

to leading order (see appendix A). In particular, for some $s > 0$ sufficiently small, $\partial \overline{B}(p, s)$ has negative mean curvature (with respect to $\tilde{g}$) in direction pointing away from p. Let $\tilde{\Omega}$ be $\Omega \setminus B(p, s)$, and restrict $\tilde{g}$ to $\tilde{\Omega}$.

The manifold $(\tilde{\Omega}, \tilde{g})$ has boundary with two connected components, both of positive mean curvature in the *outward* direction. By Lemma 6 below, $(\tilde{\Omega}, \tilde{g})$ contains a subset that is a valid fill-in of (Σ, γ, H) and has a minimal boundary component. This contradicts the assumption that the Bartnik data is of zero type. □

To complete the proof of the previous proposition, we have the following lemma:

Lemma 6. *Suppose $\mathcal{B} = (\Sigma, \gamma, H)$ admits a fill-in (Ω, g) with nonnegative scalar curvature, such that $\partial \Omega \setminus \Sigma$ has positive mean curvature in the outward direction. Then a subset Ω' of Ω is a valid fill-in of $\mathcal{B}$ with metric $g|_{\Omega'}$. Moreover, Ω' has at least one minimal boundary component.*

Proof. By assumption Σ has positive mean curvature H and $\partial \Omega \setminus \Sigma$ has positive mean curvature. By Theorem 18 in appendix B, there exists a smooth, embedded minimal surface S homologous to Σ. The closure of the region bounded between Σ and S is the desired valid fill-in. □

2.2. Data of positive type.

Proposition 7. *Bartnik data $\mathcal{B}$ is of positive type if and only if $\mathcal{B}$ admits a valid fill-in that has positive scalar curvature at some point.*

The idea of proving the proposition is to create positive energy density at some interior points at the expense of decreasing the size of the minimal surface. As in the previous section, the delicate issue is preserving the boundary data in the process.

Proof. If $\mathcal{B}$ admits a valid fill-in with positive scalar curvature at a point, then $\mathcal{B}$ is of nonnegative type and Proposition 5 rules out the case of zero type (since static vacuum metrics have zero scalar curvature).

On the other hand, suppose $\mathcal{B} = (\Sigma, \gamma, H)$ has positive type, so there exists some valid fill-in (Ω, g) of $\mathcal{B}$ with boundary $\Sigma \dot\cup S$, where S is a nonempty minimal surface. If (Ω, g) is not static vacuum, we may complete the proof by again using the work of Corvino to perturb (Ω, g) to a valid fill-in with positive scalar curvature at a point [11]. Thus, assume (Ω, g) is static vacuum, and so is in particular scalar-flat.

Replace (Ω, g) with its double across the minimal surface S. Now, (Ω, g) has two boundary components Σ and Σ' (its reflected copy), and contains a minimal surface S that is fixed by the $\mathbb{Z}_2$ reflection symmetry. Moreover, g is Lipschitz continuous across S and smooth elsewhere[1]. For simplicity of exposition, we separately treat the cases in which g is smooth and non-smooth across S.

Smooth case: For $\epsilon \in (0, 1)$, let φ be the function on Ω solving the following Dirichlet problem:

$$\begin{cases} \Delta\varphi = 0 & \text{on } \Omega \\ \varphi = 1 & \text{on } \Sigma \\ \varphi = 1 - \epsilon & \text{on } \Sigma' \end{cases}$$

Consider the conformal metric $\tilde{g} = \varphi^4 g$, which is smooth with zero scalar curvature. Moreover, the mean curvature $\tilde{H}$ of Σ with respect to $\tilde{g}$ strictly exceeds H (for all choices of ϵ), since φ has positive outward normal derivative on Σ (see appendix A). The mean curvature of Σ' is diminished, but remains positive for $\epsilon > 0$ sufficiently small. Fix such an ϵ. This construction is demonstrated in figure 2.

FIGURE 2. Construction in proof of Proposition 7
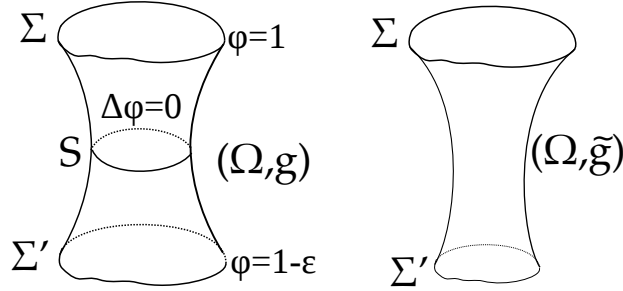
On the left is the double of (Ω, g), which we also refer to as (Ω, g), abusing notation. The function φ is harmonic, with the given prescribed Dirichlet boundary values. On the right is Ω equipped with the metric $\tilde{g}$, obtained from g by applying the conformal factor φ^4.

Fix any smooth function $\rho > 0$ on Ω. For all $\delta \geq 0$ small, let u_δ be the unique solution to the elliptic problem

$$\begin{cases} \tilde{L}u_\delta = \delta\rho & \text{in } \tilde{\Omega} \\ u_\delta = 1 & \text{on } \Sigma \\ \partial_\nu(u_\delta) = 0 & \text{on } \Sigma', \end{cases}$$

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[1]This doubling trick across a minimal surface was used by Bunting and Masood-ul-Alam to classify static vacuum metrics with compact minimal boundary that are asymptotically flat [10]. Because of the last condition, their theorem does not apply to the present case. We also mention the fact that because of minimality and the static vacuum condition, S is totally geodesic, which implies that $\tilde{g}$ is $C^{1,1}$ across S [10, 11]. However, we do not need this fact.

where $\tilde{L} = -8\tilde{\Delta}$ is the conformal Laplacian of $\tilde{g}$. Clearly $u_0 \equiv 1$, and u_δ converges in C^2 to 1 as $\delta \to 0^+$. The conformal metric $u_\delta^4 \tilde{g}$ has the following properties:

- positive scalar curvature (equal to $\delta \rho u_\delta^{-5}$),
- the induced metric on Σ equals γ (by the boundary condition $u_\delta|_\Sigma = 1$),
- positive mean curvature on Σ' (for $\delta > 0$ small enough),
- mean curvature on Σ converging uniformly to $\tilde{H}$ as $\delta \to 0^+$.

Fix a particular value of δ such that the mean curvature of Σ' is positive and the mean curvature $\tilde{H}_\delta$ of Σ is pointwise greater than H (which is possible, since $\tilde{H} > H$). By Lemma 6, there is a valid fill-in of $(\Sigma, \gamma, \tilde{H}_\delta)$ that contains a minimal surface. By Lemma 19 in appendix C, this valid fill-in can be perturbed to a valid fill-in of (Σ, γ, H) so that the latter fill-in still has positive scalar curvature at some point.

Lipschitz case: In general we must carry out an extra step to deal with the lack of smoothness across S. Define φ analogously by first solving $\Delta\varphi_1 = 0$ with boundary conditions of 1 on Σ and $1 - \frac{\epsilon}{2}$ on S, then defining $\varphi_2 = 2 - \epsilon - \varphi_1$ in the reflected copy. The function φ obtained by gluing φ_1 and φ_2 is $C^{1,1}$ on Ω, and smooth and harmonic away from S. Again, let $\tilde{g} = \varphi^4 g$, which has zero scalar curvature (away from S), is Lipschitz across S, and induces the same mean curvature on both sides of S. Fix $\epsilon > 0$ so that $\tilde{H} > H$ and the $\tilde{g}$-mean curvature of Σ' is positive.

By the work of Miao [19], the fact that both sides of S have the same mean curvature implies the existence of a family of C^2 metrics $\{\tilde{g}_\delta\}_{0 < \delta < \delta_0}$ such that

(1) $\tilde{g}_\delta$ converges to $\tilde{g}$ in C^0 as $\delta \to 0$,
(2) $\tilde{g}_\delta$ agrees with $\tilde{g}$ outside a δ-neighborhood of S, and
(3) the scalar curvature $\tilde{R}_\delta$ of $\tilde{g}_\delta$ is bounded below by a constant independent of δ.

In particular, the L^p norm of $\tilde{R}_\delta$ (taken with respect to $\tilde{g}$ or $\tilde{g}_\delta$) for any $1 \le p < \infty$ converges to zero as $\delta \to 0$. Taking $p = \frac{3}{2}$ and using similar arguments to Schoen and Yau [22], for each $\delta > 0$ sufficiently small the conformal Laplacian of $\tilde{g}_\delta$, $\tilde{L}_\delta = -8\tilde{\Delta}_\delta + \tilde{R}_\delta$, has trivial kernel on functions v with boundary conditions of $v = 0$ on Σ and $\partial_\nu v = 0$ on Σ'. Fix a smooth function $\rho > 0$ on Ω. For all $\delta > 0$ small, let u_δ be the unique solution to the problem:

$$\begin{cases} \tilde{L}_\delta u_\delta = \delta\rho & \text{in } \tilde{\Omega} \\ u_\delta = 1 & \text{on } \Sigma \\ \partial_\nu(u_\delta) = 0 & \text{on } \Sigma' \end{cases}$$

A key fact is that u_δ converges to 1 in C^0 as $\delta \to 0^+$, and this convergence is C^2 away from S (see the proof of Proposition 4.1 of Miao [19]).

At this point, the proof follows nearly the same steps as in the smooth case, where we work with the metric $u_\delta^4 \tilde{g}_\delta$ (which has positive scalar curvature and induces the metric γ on Σ). We pick $\delta > 0$ sufficiently small so that $\tilde{H}_\delta > H$ and Σ' has positive mean curvature with respect to $u_\delta^4 \tilde{g}_\delta$. Now, if necessary, perturb the C^2 metric $u_\delta^4 \tilde{g}_\delta$ on a neighborhood of S to a C^∞ metric, preserving the above properties. The proof now goes as in the smooth case, making use of Lemmas 6 and 19. $\square$

2.3. Bartnik inner mass.

One source of inspiration for the problem of considering valid fill-ins with minimal boundary is Bray's definition of the Bartnik inner mass [8], an example of a quasi-local mass (see section 4 for more on quasi local mass). The Bartnik inner mass aims to measure the size of the largest black hole that could be placed inside a valid fill-in of given Bartnik data.

Definition 8. The **Bartnik inner mass** of Bartnik data $\mathcal{B}$ is the real number

$$m_{inner}(\mathcal{B}) = \sup_{(\Omega,g)} \left\{ \sqrt{\frac{A}{16\pi}} \right\}$$

where the supremum is taken over the class of all valid fill-ins (Ω, g) of $\mathcal{B}$, and A is the minimum area in the homology class of Σ in (Ω, g).

This definition, though formulated differently, is equivalent to Bray's. The purpose of using the minimum area in the homology class of Σ is to ignore any large minimal surfaces "hidden behind" a smaller minimal surface.

We observe that the sign of $m(\mathcal{B})$ corresponds directly to the type of the Bartnik data $\mathcal{B}$. To see this, first note that for fill-ins with a minimal boundary, the minimum area of A in the homology class of Σ in (Ω, g) is always attained by a smooth minimal surface, and so A is positive (see Theorem 18). For fill-ins without boundary, Σ is homologically trivial, and so $A = 0$. Then:

- $m_{inner}(\mathcal{B}) = -\infty$ if and only if there are no valid fill-ins.
- $m_{inner}(\mathcal{B}) > 0$ if and only if there exist fill-ins for which A is positive.
- $m_{inner}(\mathcal{B}) = 0$ if and only if every fill-in has minimum area of zero in the homology class on Σ, which occurs if and only if Σ is homologically trivial in every valid fill-in, which occurs if and only if $\mathcal{B}$ is of zero type by the proof of Proposition 5.

3. THE INTERVAL OF POSITIVITY

The following idea was suggested by Bray: as a function of a parameter $\lambda > 0$, consider the Bartnik data $(\Sigma, \gamma, \lambda H)$. The primary purpose of this section is to state and prove Theorem 10, which says roughly that this data moves among all three types (positive, zero, and negative) as λ varies.

One key ingredient is the following well-known theorem of Shi and Tam.

Theorem 9 (Shi–Tam, 2002 [23]). *Fix Bartnik data (Σ, γ, H) that possesses a valid fill-in (Ω, g) and assume γ has positive Gauss curvature. Then:*

$$\int_\Sigma (H_0 - H) dA_\gamma \geq 0 \tag{3}$$

where H_0 is the mean curvature of an isometric embedding of (Σ, γ) into Euclidean space $\mathbb{R}^3$, and dA_γ is the area form on Σ with respect to the metric γ. Moreover, equality holds if and only if (Ω, g) is isometric to a subdomain of $\mathbb{R}^3$.

Note that H_0 is well-defined, since an isometric embedding of a positive Gauss curvature surface into $\mathbb{R}^3$ is unique up to rigid motions (see the references in [23]).

In our case, inequality (3), which depends only on the Bartnik data, must be satisfied for data that admits a valid fill-in. In particular, by increasing H (while keeping γ, and therefore H_0, fixed), it is clear that some Bartnik data do not possess fill-ins (i.e., are of negative type). Hence, the Shi–Tam theorem gives an obstruction to Bartnik data being of nonnegative type.

The following main theorem demonstrates that there exists a unique interval of values of λ for which this data $(\Sigma, \gamma, \lambda H)$ is of positive type.

Theorem 10. *Fix Bartnik data (Σ, γ, H). There exists a unique number $\lambda_0 > 0$ such that $(\Sigma, \gamma, \lambda H)$ is of positive type if and only if $\lambda \in (0, \lambda_0)$. Moreover, $(\Sigma, \gamma, \lambda H)$ is of negative type if $\lambda > \lambda_0$.*

As a consequence, $(\Sigma, \gamma, \lambda H)$ is zero type for at most one value of λ, namely λ_0. A further discussion follows the proof.

Proof. Define

$$I_+ = \{\lambda \in \mathbb{R}^+ : (\Sigma, \gamma, \lambda H) \text{ is of positive type}\},$$
$$I_0 = \{\lambda \in \mathbb{R}^+ : (\Sigma, \gamma, \lambda H) \text{ is of zero type}\},$$
$$I_{\geq 0} = I_+ \cup I_0$$

Step 1: We first show I_+ is nonempty. Consider the space $\Omega = \Sigma \times [-1, 0]$ with product metric g, and identify Σ with $\Sigma \times \{0\}$. Let S be the other boundary component of Ω, namely $\Sigma \times \{-1\}$. Observe that 1) Ω has positive scalar curvature since Σ has positive Gauss curvature, and 2) all leaves $\Sigma \times \{t\}$ are minimal surfaces.

Choose a smooth function v on Ω satisfying the following properties: $v \leq 0$, v vanishes on Σ and in a neighborhood of S, and $\partial_\nu v = \frac{1}{4} H$ on Σ. For $\epsilon > 0$, let $u_\epsilon = 1 + \epsilon v$. In particular, u_ϵ is positive for $\epsilon > 0$ sufficiently small. Consider the conformal metric $g_\epsilon = u_\epsilon^4 g$. Note that g_ϵ induces the metric γ on Σ, and assigns the following value to the mean curvature of Σ:

$$H_\epsilon = 4\partial_\nu(u_\epsilon) = 4\epsilon\partial_\nu v = \epsilon H,$$

by our choice of v. Moreover, the scalar curvature of g_ϵ is

$$R_{g_\epsilon} = u_\epsilon^{-5}\left(-8\Delta_g u_\epsilon + R_g u_\epsilon\right)$$
$$= u_\epsilon^{-5}\left(-8\epsilon\Delta_g v + R_g u_\epsilon\right),$$

which is positive for ϵ sufficiently small, since u_ϵ and R_g are uniformly bounded below as $\epsilon \to 0^+$. Fix such an ϵ. We can see (Ω, g_ϵ) is a valid fill-in of $(\Sigma, \gamma, \epsilon H)$, since this fill-in has positive scalar curvature, induces the correct boundary geometry on Σ, and S is minimal (since $g_\epsilon = g$ near S). In particular, ϵ belongs to I_+, so $I_+ \neq \varnothing$.

Step 2: The next step is to show that $I_{\geq 0}$ is connected, and I_0 contains at most one point. To accomplish this, we show that for every number in $I_{\geq 0}$, every smaller positive number belongs to I_+. It suffices to show that if (Σ, γ, H) is of nonnegative type, then $(\Sigma, \gamma, \lambda H)$ is of positive type for all $\lambda \in (0, 1)$. Take a valid fill-in (Ω, g) of the former. Working in a neighborhood $\Sigma \times (-t_0, 0]$ of $\Sigma \subset \Omega$, we may assume g takes the form

$$g = dt^2 + G_t,$$

where t is the g-distance to Σ, and G_t is a Riemannian metric on the surface $\Sigma_t = \Sigma \times \{t\}$. Shrinking t_0 if necessary, we may assume that every (Σ_t, G_t) has positive Gauss curvature K_t and positive mean curvature H_t (in the outward direction ∂_t). Let $\rho : (-t_0, 0] \to \mathbb{R}$ be a smooth function satisfying

(1) $\rho \equiv 1$ in a neighborhood of $-t_0$,
(2) $\rho(0) = \lambda^{-1} \geq 1$,
(3) $\rho'(t) \geq 0$, and
(4) $\rho'(0) > 0$.

Define a new metric $\tilde{g}$ on Ω by setting

$$\tilde{g} = \rho(t)^2 dt^2 + G_t \tag{4}$$

on the neighborhood of Σ, and extending smoothly by g to the rest of Ω. A straightforward calculation shows that Σ has mean curvature λH in the metric $\tilde{g}$; moreover $\tilde{g}$ induces the metric γ on Σ. Also, note that the surface $\partial\Omega \setminus \Sigma$ (if nonempty) is minimal in $\tilde{g}$, since $\tilde{g} = g$ near $\partial\Omega \setminus \Sigma$. We will have shown $(\Sigma, \gamma, \lambda H)$ has valid fill-in $(\Omega, \tilde{g})$ once we verify that the latter has nonnegative scalar curvature. This need only be checked in the neighborhood $\Sigma \times (-t_0, 0]$. From the second variation of area formula, we have the following expression for the scalar curvature of g:

$$R_g = -2\frac{\partial H_t}{\partial t} + 2K_t - H_t^2 + \|h_t\|^2, \tag{5}$$

where h_t is the second fundamental form of Σ_t in (Ω, g), and its norm $\|\cdot\|^2$ is taken with respect to G_t. Applying this formula to the metric $\tilde{g}$ yields

$$R_{\tilde{g}} = \frac{1}{\rho(t)^2}R_g + 2K_t(1 - \rho(t)^{-2}) + 2\frac{\rho'(t)}{\rho(t)^3}H_t. \tag{6}$$

From this, we see that $R_{\tilde{g}} \geq 0$, since $R_g \geq 0, K_t > 0, \rho(t) \geq 1, \rho'(t) \geq 0$ and $H_t > 0$. The data $(\Sigma, \gamma, \lambda H)$ has a valid fill-in $(\Omega, \tilde{g})$ with positive scalar curvature in a neighborhood of Σ. By Proposition 7, $(\Sigma, \gamma, \lambda H)$ is of positive type.

We conclude that $I_{\geq 0}$ is a convex subset of $\mathbb{R}^+$, containing all arbitrarily small positive numbers. Moreover, I_0 contains at most a single point.

Step 3: We prove that $I_{\geq 0}$ is bounded above. This follows immediately from the work of Shi and Tam. More precisely, if $\lambda \in I_{\geq 0}$, then

$$\lambda \leq \frac{\int_\Sigma H_0 dA_\gamma}{\int_\Sigma H dA_\gamma}.$$

Together with step 2, we see $I_{\geq 0}$ and I_+ are intervals of the form $(0, \lambda_0]$ or $(0, \lambda_0)$.

Step 4: Here we prove that λ_0 does not belong to I_+. If $\lambda_0 \in I_+$, then by Proposition 7, there exists a valid fill-in (Ω, g) of $(\Sigma, \gamma, \lambda_0 H)$ with positive scalar curvature at some point and boundary $\Sigma \dot\cup S_0$, with S_0 minimal and nonempty. Solve the mixed Dirichlet–Neumann problem:

$$\begin{cases} \Delta u = \frac{1}{8}R_g u & \text{in } \Omega \\ u = 1 & \text{on } \Sigma \\ \partial_\nu(u) = 0 & \text{on } S_0. \end{cases} \tag{7}$$

Here, ν is the unit normal, always chosen to point out of Ω. Note that a solution exists because $R_g \geq 0$. By the maximum principle, $u > 0$ in Ω and $\partial_\nu(u) > 0$ on Σ. Let $g' = u^4 g$. Note that g' has zero scalar curvature, induces the metric γ on Σ and assigns zero mean curvature to S_0. In particular, if we let H' be the mean curvature of Σ with respect to g', then (Σ, γ, H') has a valid fill-in with minimal boundary, namely (Ω, g'), and is therefore of positive type. Observe that $H' > \lambda_0 H$. Choose $\beta > 1$ so that $H' > \beta\lambda_0 H$. By Lemma 20 in appendix C, we see that $(\Sigma, \gamma, \beta\lambda_0 H)$ is of positive type. Therefore $\beta\lambda_0 \in I_+$, which contradicts $\lambda_0 = \sup I_+$. We conclude $I_+ = (0, \lambda_0)$, and either $I_{\geq 0} = (0, \lambda_0)$ or $(0, \lambda_0]$. It follows that if $\lambda > \lambda_0$, then $(\Sigma, \gamma, \lambda H)$ must be of negative type.

$\square$

To emphasize the picture, the data $(\Sigma, \gamma, \lambda H)$ is of positive type for λ small. As we increase λ, this behavior persists until $\lambda = \lambda_0$. At this point, the data is zero or negative, and for $\lambda > \lambda_0$, the data is negative. However, we conjecture:

Conjecture 11. *The Bartnik data $(\Sigma, \gamma, \lambda_0 H)$ is of zero type.*

While this conjecture may appear benign, we see from Proposition 5, that an affirmative answer would immediately imply that $(\Sigma, \gamma, \lambda_0 H)$ admits a "static vacuum fill-in." In general, constructing static vacuum metrics with prescribed boundary data is a very difficult problem (c.f. the work of Anderson and Khuri on static vacuum asymptotically flat "extensions" of Bartnik data [2]).

3.1. **Inner mass function.** In the remainder of this section we will study the function

$$m(\lambda) = m_{inner}(\Sigma, \gamma, \lambda H) \tag{8}$$

defined for $\lambda \in (0, \lambda_0)$. Intuitively, one would expect the following behavior of the function $m(\lambda)$. For λ very small, the mean curvature λH is very close to zero, so one might anticipate the existence of a valid fill-in with minimal boundary of approximately the same area as Σ.

As λ increases, one would expect the class of valid fill-ins to shrink; one reason is that the Shi–Tam inequality is more difficult to satisfy. Consequently, the Bartnik inner mass ought to decrease as well. The following theorem supports this intuition.

Theorem 12. *Given Bartnik data (Σ, γ, H), the function $m : (0, \lambda_0) \to \mathbb{R}^+$ is continuous and decreasing, with the following limiting behavior:*

$$\lim_{\lambda \to 0^+} m(\lambda) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}.$$

Here, $|\Sigma|_\gamma$ is the area of Σ with respect to γ.

Proof. Monotonicity: Given $0 < \lambda_1 < \lambda_2 < \lambda_0$, we showed in Step 2 of the proof of Theorem 10 that any valid fill-in of $(\Sigma, \gamma, \lambda_2 H)$ gives rise to a valid fill-in of $(\Sigma, \gamma, \lambda_1 H)$ with a metric that is pointwise larger (see (4)). From the definition of the Bartnik inner mass, this shows that

$$m(\lambda_1) \geq m(\lambda_2).$$

Continuity: Suppose $0 < \lambda_1 < \lambda_0$, and let $\epsilon > 0$. From the definition of the Bartnik inner mass, there exists a valid fill-in (Ω, g) of $(\Sigma, \gamma, \lambda_1 H)$ whose minimum area A in the homology class of Σ satisfies

$$m(\lambda_1) - \sqrt{\frac{A}{16\pi}} < \frac{\epsilon}{3}.$$

From the proof of Proposition 7, it is clear that there exists a valid fill-in $(\tilde{\Omega}, \tilde{g})$ of $(\Sigma, \gamma, \lambda_1 H)$ that has strictly positive scalar curvature, and whose minimum area $\tilde{A}$ in the homology class of Σ is close to A:

$$\sqrt{\frac{A}{16\pi}} - \sqrt{\frac{\tilde{A}}{16\pi}} < \frac{\epsilon}{3}.$$

Now, for $\lambda > \lambda_1$, $(\tilde{\Omega}, \tilde{g})$ can be perturbed to a fill-in $(\tilde{\Omega}, \tilde{g}_\lambda)$ of $(\Sigma, \gamma, \lambda H)$ using a metric of the form (4). The scalar curvature of $\tilde{g}_\lambda$ has potentially decreased relative to that of $\tilde{g}$, but remains positive for $\lambda > \lambda_1$ sufficiently close to λ. Since $\tilde{g}_\lambda \to \tilde{g}$ in C^0, we may assume $\lambda - \lambda_1$ is small enough so that

$$\sqrt{\frac{\tilde{A}}{16\pi}} - \sqrt{\frac{\tilde{A}_\lambda}{16\pi}} < \frac{\epsilon}{3},$$

where $\tilde{A}_\lambda$ is the minimum $\tilde{g}_\lambda$-area in the homology class of Σ. Adding the last three inequalities and using the definition of the Bartnik inner mass gives

$$m(\lambda_1) < \epsilon + \sqrt{\frac{\tilde{A}_\lambda}{16\pi}}$$
$$\leq \epsilon + m(\lambda)$$

for $\lambda - \lambda_1$ sufficiently small. Together with the fact that $m(\cdot)$ is decreasing, we have shown $m(\cdot)$ is continuous at λ_1.

Lower limit behavior: To study the behavior of $m(\epsilon)$ for ϵ small, recall that in Step 1 of the proof of Theorem 10 we constructed a valid fill-in of $(\Sigma, \gamma, \epsilon H)$ by a metric g_ϵ uniformly close (controlled by ϵ) to a cylindrical product metric g over (Σ, γ). As $\epsilon \to 0^+$, the minimum g_ϵ area in the homology class of Σ converges to the minimum g-area in the same homology class, which is $|\Sigma|_\gamma$. On the other hand, the Bartnik inner mass of (Σ, γ, H') (for any H') never exceeds $\sqrt{\frac{|\Sigma|_\gamma}{16\pi}}$ by definition. This proves

$$\lim_{\lambda \to 0+} m(\lambda) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}.$$

$\square$

In section 7 we conjecture that $m(\lambda)$ limits to zero as $\lambda \to \lambda_0^-$, behavior supported by the explicit computation of $m(\lambda)$ in a spherically-symmetric case in section 5.1.

4. Quasi-local mass

Recall from the introduction the problem of assigning a "quasi-local mass" to a bounded region Ω in a totally geodesic spacelike slice (M, g) of a spacetime. By most definitions, the quasi-local mass of Ω depends only on the Bartnik data (Σ, γ, H) of the boundary, and we adopt this perspective here. That is, we define a *quasi-local mass functional* to be a map from (a subspace of) the set of Bartnik data to the real numbers. We refer the reader to [24] for a recent comprehensive survey of quasi-local mass.

We begin by recalling some well-known examples of quasi-local mass. First, the *Hawking mass* of (Σ, γ, H) is defined to be

$$m_H(\Sigma, \gamma, H) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(1 - \frac{1}{16\pi} \int_\Sigma H^2 dA_\gamma\right).$$

There is no correlation between the sign of the Bartnik data and the sign of the Hawking mass. That is, the Hawking mass can be negative for positive Bartnik data, and vice versa (see section 5).

Next, the *Brown–York mass* is defined for Bartnik data (Σ, γ, H) (assuming as we do that $K_\gamma > 0$ and $H > 0$) by

$$m_{BY}(\Sigma, \gamma, H) = \frac{1}{8\pi} \int_\Sigma (H_0 - H) dA_\gamma,$$

where H_0 is the mean curvature of an isometric embedding of (Σ, γ) into $\mathbb{R}^3$. Theorem 9 of Shi–Tam establishes that the Brown–York mass is nonnegative for Bartnik data of nonnegative type. However, there exist Bartnik data of both negative and zero type for which the Brown–York mass is strictly positive (see section 5).

A third example is the Bartnik inner mass, defined in section 2.3.

A key observation is that Theorem 10 canonically associates to any Bartnik data (with $H > 0$ and $K_\gamma > 0$) a positive number λ_0, which we call the *critical parameter*. In this section we use λ_0 to construct a new example of a quasi-local mass functional.

To motivate this definition, we will compute the number λ_0 for concentric round spheres Σ_r in the Schwarzschild manifold of mass m, with induced metric γ_r and mean curvature H_r. For our purposes the Schwarzschild manifold of mass m is $\mathbb{R}^3$ minus the open Euclidean ball of radius $m/2$, where $m > 0$, equipped with the metric

$$g = \left(1 + \frac{m}{2r}\right)^4 \delta, \tag{9}$$

where δ is the Euclidean metric. Note that g is scalar-flat and its boundary is a minimal 2-sphere, called the horizon.

Straightforward computations show that (Σ_r, γ_r) is a round sphere of area $4\pi r^2 \left(1 + \frac{m}{2r}\right)^4$, and

$$H_r = \frac{2}{r} \left(1 + \frac{m}{2r}\right)^{-2} - \frac{2m}{r^2} \left(1 + \frac{m}{2r}\right)^{-3}.$$

The mean curvature H_r^0 of (Σ_r, γ_r) embedded in $\mathbb{R}^3$ is

$$H_r^0 = \frac{2}{r} \left(1 + \frac{m}{2r}\right)^{-2}.$$

Therefore, if we let $\lambda_r = H_r^0/H_r$, then $(\Sigma_r, \gamma_r, \lambda H_r)$ admits a valid fill-in – namely a closed ball in flat-space of boundary area $4\pi r^2 \left(1 + \frac{m}{2r}\right)^4$. On the other hand, if λ belongs to the

interval of positivity for $(\Sigma_r, \gamma_r, \lambda H_r)$, then by Shi–Tam

$$\lambda \leq \frac{\int_\Sigma H_r^0 dA_\gamma}{\int_\Sigma H_r dA_\gamma} = \lambda_r.$$

Thus, λ_r is the critical parameter for the Bartnik data. Some simplifications show

$$\lambda_r = \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}.$$

In particular, we have the identity in Schwarzschild space:

$$m = \sqrt{\frac{|\Sigma_r|_g}{16\pi}} \left(1 - \frac{1}{\lambda_r^2}\right),$$

for all values of r. This identity directly motivates the following definition of quasi-local mass.

Definition 13. Let $\mathcal{B} = (\Sigma, \gamma, H)$ be Bartnik data with critical parameter λ_0 (from Theorem 10). Define

$$m(\mathcal{B}) = m(\Sigma, \gamma, H) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(1 - \frac{1}{\lambda_0^2}\right).$$

Recall that we assume γ has positive Gauss curvature and $H > 0$.

Theorem 14. *Definition 13 of quasi-local mass satisfies the following properties:*

(1) *(nonnegativity) If Bartnik data $\mathcal{B}$ admits a valid fill-in, then its mass $m(\mathcal{B})$ is non-negative and is zero only if every valid fill-in is static vacuum.*

(2) *(spherical symmetry) If Bartnik data $\mathcal{B}$ arises from a coordinate sphere in a Schwarzschild metric of mass m, then $m(\mathcal{B}) = m$.*

(3) *(black hole limit). If $\mathcal{B}_n = (\Sigma, \gamma, H_n)$ is a sequence of Bartnik data and $H_n \to 0$ uniformly, then*

$$\lim_{n\to\infty} m(\mathcal{B}_n) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}.$$

(4) *(ADM-sub-limit) If (M, g) is an asymptotically flat manifold with nonnegative scalar curvature, and if S_r is a coordinate sphere of radius r with induced metric γ_r and mean curvature H_r, then*

$$m_{ADM}(M, g) \geq \limsup_{r\to\infty} m(S_r, \gamma_r, H_r). \tag{10}$$

Remarks. The proof of Theorem 14 uses the positive mass theorem [22] implicitly, via Lemma 15 below, which relies on the theorem of Shi–Tam. On the other hand Theorem 14 also recovers the positive mass theorem: if (M, g) is asymptotically flat, has nonnegative scalar curvature, with ∂M empty or consisting of minimal surfaces, then by property (1), $m(S_r) \geq 0$ for all S_r. From this, inequality (10) gives $m_{ADM} \geq 0$.

Proof. Nonnegativity: Observe $m(\mathcal{B}) > 0$ if and only if $\lambda_0 > 1$ if and only the number 1 belongs to the interval of positivity I_+, if and only if the data is positive type. (Here, we used Theorem 10.) Also, if (Σ, γ, H) is of zero type, then $\lambda_0 = 1$ (as follows from Theorem 10), so $m(\mathcal{B})$ vanishes. On the other hand, if $m(\mathcal{B})$ vanishes, then $\lambda_0 = 1$, so the data is either negative or zero (again, by Theorem 10). But if it is given that the data admits a fill-in, then the data must be of zero type. By Proposition 5, any such fill-in is static vacuum.

Spherical symmetry: This is clear from the construction at the beginning of this section; we defined quasi-local mass so that it has this property.

Black hole limit: It is straightforward to check that if $H_n \to 0$ uniformly, then the sequence of critical parameters λ_n diverges to infinity.

ADM-sub-limit: For all r sufficiently large, the coordinate spheres S_r have positive mean and Gauss curvatures. To prove (10), recall that the Brown–York mass limits to the ADM mass in the sense that

$$m_{ADM}(M, g) = \lim_{r \to \infty} m_{BY}(S_r).$$

(See Theorem 1.1 of [12] and the references therein.) Since we assume (M, g) has nonnegative scalar curvature, S_r is of positive or zero type for all r for which the coordinate sphere is defined. We invoke Lemma 15 below, which states $m(S_r) \le m_{BY}(S_r)$, completing the proof.

$\square$

Lemma 15. *For Bartnik data $\mathcal{B} = (\Sigma, \gamma, H)$ of nonnegative type,*
$$m(\mathcal{B}) \le m_{BY}(\mathcal{B}).$$

Proof. Let H_0 be the mean curvature of an isometric embedding of (Σ, γ) in $\mathbb{R}^3$, which is well-defined because $K_\gamma > 0$. By Shi–Tam, we have $\lambda_0 \le \frac{\int_\Sigma H_0 dA_\gamma}{\int_\Sigma H dA_\gamma}$. In particular,

$$
\begin{aligned}
m(\mathcal{B}) &= \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(1 - \frac{1}{\lambda_0^2} \right) \\
&\le \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(1 - \left(\frac{\int_\Sigma H dA_\gamma}{\int_\Sigma H_0 dA_\gamma} \right)^2 \right) \\
&\le \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \left(\frac{\left(\int_\Sigma H_0 + \int_\Sigma H \right) \left(\int_\Sigma H_0 - \int_\Sigma H \right)}{\left(\int_\Sigma H_0 \right)^2} \right) \\
&\le \sqrt{\frac{|\Sigma|_\gamma}{16\pi}} \frac{16\pi m_{BY}(\mathcal{B})}{\int_\Sigma H_0},
\end{aligned}
\tag{11}
$$

where again we have used Shi–Tam and the fact that the data is of nonnegative type. The Minkowski inequality for convex regions in $\mathbb{R}^3$ [21] states that

$$\left(\int_\Sigma H_0 dA_\gamma \right)^2 \ge 16\pi |\Sigma|_\gamma.$$

Together with the above, this completes the proof. $\square$

The right-hand side of (11) is a definition of quasi-local mass proposed by Miao, which he observed is bounded above by the Brown–York mass using the same argument [18].

Remarks. Definition 13 incorporates aspects of the Hawking mass, Brown–York mass, and Bartnik inner mass. Its formula is very similar to the Hawking mass, with the scale–invariant terms $\frac{1}{\lambda_0^2}$ and $\frac{1}{16\pi} \int_\Sigma H^2 dA_\gamma$ swapped. The similarity with the Hawking mass is explained in section 6.

4.1. **Physical remarks.** It has been suggested in the literature (see [6] for instance) that if the quasi-local mass of the boundary of a region Ω vanishes, then Ω ought to be flat. The Brown–York mass and Bartnik mass both satisfy this property (see [4, 16]). Definition 13 suggests an alternative viewpoint that such Ω ought to be *static vacuum*, which includes flat metrics as a special case. Indeed, one could make a physical argument that in a region of a spacetime that is static vacuum, quasi-local mass should vanish since there is no matter content and no gravitational dynamics (c.f. [1], which also discusses the vanishing of quasi-local mass on static vacuum regions).

5. Examples

Let (M, g) be a Riemannian 3-manifold. If Ω is a subset of M with boundary $\partial\Omega$ homeomorphic to S^2, and if $\partial\Omega$ has positive mean curvature H (with respect to some chosen normal direction), define

$$m(\Omega) = m(\partial\Omega, g_{T\partial\Omega}, H),$$

where $T\partial\Omega$ is the tangent bundle of $\partial\Omega$. If g has nonnegative scalar curvature, then $m(\Omega) \geq 0$ by Theorem 14.

Flat space: Consider $\mathbb{R}^3$ with the flat metric. Let $\Omega \subset \mathbb{R}^3$ be a strictly convex open set with smooth boundary that is not round. There is no valid fill-in of $\partial\Omega$ with greater mean curvature; this statement follows from the Shi–Tam inequality (3) or alternatively by Miao's "positive mass theorem with corners" [19]. This implies $\lambda_0 = 1$, and so $m(\Omega) = 0$. The Brown–York mass of Ω also vanishes, as $H_0 = H$. A straightforward computation shows that the Hawking mass of Ω is strictly negative.

Schwarzschild, positive mass: Next let (M, g) be a Schwarzschild manifold of mass $m > 0$ (see equation (9)). Suppose $\Omega \subset M$ is topologically an open 3-ball with boundary Σ disjoint from the horizon.

Lemma 16. *For the Bartnik data induced on $\partial\Omega$, $\lambda_0 = 1$. Equivalently, $m(\Omega) = 0$.*

Figure 3 gives a depiction of the Bartnik data in question.

Figure 3. Off-center ball in Schwarzschild
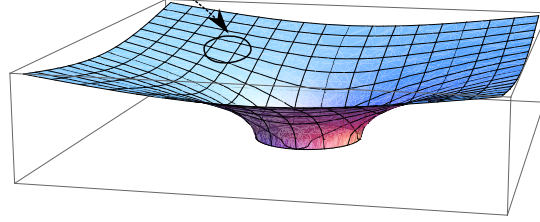
The Bartnik data (Σ, γ, H) arises from the boundary of a small ball away from the horizon in a Schwarzschild manifold.

Proof. Certainly $\lambda_0 \geq 1$, since Ω is tautologically a valid fill-in. If $\lambda_0 > 1$, there exists a valid fill-in Ω' of Σ (with the same metric and mean curvature), such that $\partial\Omega' \setminus \Sigma$ is nonempty and consists of minimal surfaces. Glue Ω' to $M \setminus \Omega$ along Σ, obtaining a manifold (M', g') that is smooth and has nonnegative scalar curvature away from Σ. Moreover, g' is Lipschitz across S, and $\partial M'$ consists of minimal surfaces (including the Schwarzschild horizon). Let A and A' be the minimum areas in the homology class of the boundary for the respective manifolds (M, g) and (M', g'). A is attained uniquely by the horizon S in M, and by a similar consideration A' is attained by a surface that includes S as a proper subset. Thus $A' > A$. By direct computation,

$$m = \sqrt{\frac{A}{16\pi}},$$

and so

$$m' < \sqrt{\frac{A'}{16\pi}}, \tag{12}$$

where $m' = m$ is the ADM mass of (M', g') (equal because g and g' agree outside a compact set). Using an argument similar to Miao [19], one can mollify (M', g') to a smooth, asymptotically flat metric of nonnegative scalar curvature and minimal boundary that gives

strict inequality in (12). This violates the well-known Riemannian Penrose inequality [8, 16]. This contradiction implies that $\lambda_0 = 1$, so $m(\Omega) = 0$. $\qquad\square$

Thus, we have examples of Bartnik data of zero type that do not arise as the boundaries of regions in flat space. In other words, we have non-flat domains Ω for which $m(\Omega) = 0$. Of course, by Theorem 14, such Ω must be static vacuum (as is the case for the Schwarzschild metric).

To further extend this example, Huisken and Ilmanen show that there exist small balls Ω away from the horizon in the Schwarzschild manifold whose Bartnik data (Σ, γ, H) have strictly positive Hawking mass. Moreover, the case of equality of Theorem 9 of Shi and Tam shows that the Brown–York mass of Ω is also strictly positive. It follows that the data $(\Sigma, \gamma, \lambda H)$ is of negative type, yet still has strictly positive m_H and m_{BY} for $\lambda > 1$ sufficiently close to 1.

Note that we have not stated Ω being static vacuum implies $m(\Omega) = 0$. Counterexamples are unknown to the author.

Schwarzschild, negative mass: Let (M, g) be the Schwarzschild metric of mass $m < 0$ (defined by (9) on $\mathbb{R}^3$ minus the closed ball of radius $|m|/2$). The Bartnik data induced on spheres $\{r = \text{const.}\}$ is of negative type because it violates the Shi–Tam inequality (3).

5.1. **Example $m(\lambda)$ function.** Here we give an explicit computation of the inner mass function $m(\lambda)$ defined in section 3.1 for Bartnik data $\mathcal{B}$ corresponding to the coordinate sphere S_r of radius $r > m/2$ in the Schwarzschild metric of mass $m > 0$, with induced metric γ and mean curvature H. The Riemannian Penrose inequality [8], shows that the Bartnik inner mass of $\mathcal{B}$ equals m. Let $\lambda > 0$; the data $(S_r, \gamma, \lambda H)$ embeds uniquely as a coordinate sphere $S_{r'}$ of some radius r' in a Schwarzschild metric of some mass m'. Equating the areas of S_r and $S_{r'}$ in their respective metrics, we have

$$4\pi r^2 \left(1 + \frac{m}{2r}\right)^4 = 4\pi (r')^2 \left(1 + \frac{m'}{2r'}\right)^4. \tag{13}$$

Equating λH with the mean curvature of $S_{r'}$ leads to

$$\lambda \left(\frac{2}{r}\left(1 + \frac{m}{2r}\right)^{-2} - \frac{2m}{r^2}\left(1 + \frac{m}{2r}\right)^{-3}\right) = \frac{2}{r'}\left(1 + \frac{m'}{2r'}\right)^{-2} - \frac{2m'}{(r')^2}\left(1 + \frac{m'}{2r'}\right)^{-3}. \tag{14}$$

With some calculations, one can compute r' and m' explicitly. For $\lambda \in (0, \lambda_0)$, we know $m(\lambda)$, the Bartnik inner mass of $S_{r'}$, simply equals m' (again, by the Riemannian Penrose inequality). Omitting some details, we give the formula:

$$m(\lambda) = \frac{r}{2}\left(\left(1 + \frac{m}{2r}\right)^2 - \lambda^2\left(1 - \frac{m}{2r}\right)^2\right). \tag{15}$$

As anticipated by Theorem 12, $m(\lambda)$ is continuous, decreasing, and $m(0) = \sqrt{\frac{A}{16\pi}}$, where A is the area of S_r in the Schwarzschild metric of mass m. Moreover, $m(\lambda)$ vanishes at the critical value $\lambda_0 = \frac{1 + \frac{m}{2r}}{1 - \frac{m}{2r}}$ (computed in section 4), a property conjectured to hold in general.

6. An algebraic operation on quasi-local mass functionals

For a quasi-local mass functional m_i (i.e., a map from the set of Bartnik data to the real numbers), define the number:

$$\lambda_i(\Sigma, \gamma, H) = \sup\{\lambda > 0 : m_i(\Sigma, \gamma, \lambda H) \geq 0\}.$$

In other words, λ_i measures how much one can scale the boundary mean curvature until the mass m_i becomes negative. Up to this point, we have studied this number for the case in which m_i is the Bartnik inner mass (since $m_{inner}(\Sigma, \gamma, H) \geq 0$ if and only if (Σ, γ, H)

has a valid fill-in). Here we use the number λ_i to construct an algebraic product of two quasi-local mass functionals, of which that constructed in section 4 is a special case.

We restrict to quasi-local mass functionals m_i satisfying the following mild assumptions on all Bartnik data:

(1) $\lambda_i(\Sigma, \gamma, H)$ is a positive real number, and
(2) $m_i(\Sigma, \gamma, \lambda H)$ is strictly decreasing as a function of λ.

Define the following binary operation on the set of quasi-local mass functionals. Given m_1 and m_2, let

$$(m_1 * m_2)(\Sigma, \gamma, H) = m_1\left(\Sigma, \gamma, \frac{\lambda_1}{\lambda_2} H\right), \tag{16}$$

where $\lambda_i = \lambda_i(\Sigma, \gamma, H)$ for $i = 1, 2$. This operation satisfies a number of properties, all of which essentially follow immediately from the definitions (and whose proofs are omitted).

Proposition 17. *Let $m_1, m_2,$ and m_3 be quasi-local mass functionals.*

(1) $m_1 * m_1 = m_1$.
(2) $(m_1 * m_2) * m_3 = m_1 * m_3 = m_1 * (m_2 * m_3)$. In particular, $*$ is associative.
(3) m_2 controls the sign of $m_1 * m_2$ in the following sense:
  (a) $m_1 * m_2(\Sigma, \gamma, H) > 0$ if and only if $m_2(\Sigma, \gamma, H) > 0$, and
  (b) $m_1 * m_2(\Sigma, \gamma, H) = 0$ if and only if $m_2(\Sigma, \gamma, H) = 0$.
(4) If m_1 has the black hole limit property (see Theorem 14), so does $m_1 * m_2$.
(5) If both m_1 and m_2 produce the value m on concentric round spheres in the Schwarzschild metric of mass m, then so does $m_1 * m_2$.
(6) If $m_2 \leq m_3$ (as functions), then $m_1 * m_2 \leq m_1 * m_3$.

In the next section, we demonstrate $m_1 * m_2$ generally does not equal $m_2 * m_1$.

6.1. **Examples of $m_1 * m_2$.**
Hawking mass and Bartnik inner mass: The quasi-local mass of Definition 13 is equal to $m_H * m_{inner}$, where m_H is the Hawking mass. To see this, note that $\lambda_H = \sqrt{\frac{16\pi}{\int_\Sigma H^2 dA_\gamma}}$ and $\lambda_{inner} = \lambda_0$. Then by definition,

$$m_H * m_{inner}(\Sigma, \gamma, H) = m_H\left(\Sigma, \gamma, \frac{\lambda_H}{\lambda_0} H\right)$$
$$= \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}\left(1 - \frac{1}{\lambda_0^2}\right).$$

We reiterate that $m_H * m_{inner}$ inherits the following property from m_{inner}: vanishing precisely on Bartnik data of zero type.

Hawking mass and Brown–York mass: To compute $m_H * m_{BY}$, we note λ_H was found in the last example, and $\lambda_{BY} = \frac{\int_\Sigma H_0 dA_\gamma}{\int_\Sigma H dA_\gamma}$. Using the definition,

$$m_H * m_{BY}(\Sigma, \gamma, H) = \sqrt{\frac{|\Sigma|_\gamma}{16\pi}}\left(1 - \left(\frac{\int_\Sigma H dA_\gamma}{\int_\Sigma H_0 dA_\gamma}\right)^2\right).$$

This quasi-local mass was written down in a different context by Miao [18].

Brown–York mass and Hawking mass: The steps from the last example show

$$m_{BY} * m_H(\Sigma, \gamma, H) = \int_\Sigma H_0 dA_\gamma\left(1 - \sqrt{\frac{\int_\Sigma H^2 dA_\gamma}{16\pi}}\right),$$

illustrating concretely the non-commutativity of $*$.

7. CONCLUDING REMARKS

We conclude by mentioning some questions raised in this paper.

Problem 1. *Determine whether the quasi-local mass of Definition 13 is monotone under some flow.*

Monotonicity means that if $\{(\Sigma_t, \gamma_t, H_t)\}_{t \in [0,\epsilon)}$ is some family of surfaces (together with their Bartnik data) moving outward in a manifold of nonnegative scalar curvature, then $m(\Sigma_t, \gamma_t, H_t)$ is non-decreasing. Monotonicity is often (but not universally) suggested as a desirable property of quasi-local mass [6].

Problem 2. *Determine whether the Bartnik data $(\Sigma, \gamma, \lambda_0 H)$ is of zero type. Equivalently, construct a static vacuum fill-in of $(\Sigma, \gamma, \lambda_0 H)$.*

That the two above statements are equivalent follows from Proposition 5 and Theorem 10. The precise nature of the Bartnik data rescaled with the critical parameter λ_0 is perhaps the biggest open question of this paper. More generally, one could ask what happens to the geometry of the class valid fill-ins of $(\Sigma, \gamma, \lambda H)$ in the limit $\lambda \nearrow \lambda_0$. A conjecture, perhaps overly optimistic, is that in the limit $\lambda \nearrow \lambda_0$, any valid fill-in $(\Omega_\lambda, g_\lambda)$ of $(\Sigma, \gamma, \lambda H)$ satisfies:

- the black holes (area-minimizing minimal surfaces) in $(\Omega_\lambda, g_\lambda)$ are shrinking to zero size, and
- the metric g_λ is "approaching" a static vacuum metric in an appropriate sense.

There may be a connection between the first point and Miao's localized Riemannian Penrose inequality [18].

The above discussion is basically a localization of the near-equality case of the positive mass theorem [22]. In such a global setting, the question is: what happens to the geometry of a sequence of asymptotically flat manifolds (M_i, g_i) of nonnegative scalar curvature whose total mass is approaching zero? The Riemannian Penrose inequality [8, 16] shows that any black holes in (M_i, g_i) must be approaching zero, and some partial results exist for proving that g_i is approaching a flat metric [5, 17].

APPENDIX A. CONFORMAL TRANSFORMATION OF CURVATURES

We repeatedly used the following formulas that relate the scalar curvature and mean curvature of conformal metrics. Suppose g and $\overline{g}$ are Riemannian metrics on a 3-manifold for which $\overline{g} = u^4 g$ for some smooth function $u > 0$. If R and $\overline{R}$ are the scalar curvatures of g and $\overline{g}$, then

$$\overline{R} = u^{-5}(-8\Delta u + Ru), \tag{17}$$

where Δ is the Laplacian with respect to g. Next, suppose S is a hypersurface with unit normal field ν with respect to g. Then the mean curvatures H and $\overline{H}$ (in the direction defined by ν) with respect to g and $\overline{g}$ satisfy:

$$\overline{H} = u^{-2}H + 4u^{-3}\nu(u). \tag{18}$$

APPENDIX B. GEOMETRIC MEASURE THEORY

Here is an extremely useful result from geometric measure theory on the existence and regularity of area-minimizing surfaces.

Theorem 18. *Let (M, g) be a smooth, compact Riemannian manifold of dimension $2 \leq n \leq 7$ with boundary ∂M. Suppose ∂M has positive mean curvature in the outward direction. Given a connected component S of ∂M, there exists a smooth, embedded hypersurface $\tilde{S}$ of zero mean curvature that minimizes area among surfaces homologous to S. Moreover, $\tilde{S}$ does not intersect ∂M.*

These results are essentially due to Federer and Fleming [13–15]. The rough idea of the proof of Theorem 18 is to take a minimizing sequence of surfaces $\{S_i\}$ (viewed as integral currents) in $[S]$, the homology class of S. By the Federer–Fleming compactness theorem, some subsequence converges to a surface $\tilde{S}$. Standard arguments show that $\tilde{S}$ remains in $[S]$ and indeed has the desired minimum of area. Regularity theory (requiring $n \leq 7$) proves that $\tilde{S}$ is a smooth, embedded hypersurface. By the first variation of area formula, $\tilde{S}$ has zero mean curvature and may not touch the positive mean curvature boundary (which acts as a barrier). See the appendix of [22] for a careful proof of the last fact.

Appendix C. Deformations of scalar curvature near a boundary

Our goal is to prove the following useful lemma.

Lemma 19. *Suppose that (Σ, γ, H_1) admits a valid fill-in. If $0 < H_2 < H_1$, then (Σ, γ, H_2) admits a valid fill-in with positive scalar curvature at a point.*

In conjunction with Proposition 7, the above immediately implies

Lemma 20. *Suppose that (Σ, γ, H_1) is of nonnegative type. If $0 < H_2 < H_1$, then (Σ, γ, H_2) is of positive type.*

The proof of Lemma 19 is an application of techniques developed recently by Brendle, Marques, and Neves [9]. Although we only prove the case $K_\gamma > 0$ here, Lemma 19 is true without this hypothesis.

Proof. Step 1: We construct a valid fill-in of (Σ, γ) with mean curvature strictly greater than H_2 and with positive scalar curvature in a neighborhood of Σ.

Since Σ is compact, we may choose $\alpha \in (0, 1)$ so that $\alpha H_1 > H_2$. We proved in step 2 of Theorem 10 that $(\Sigma, \gamma, \alpha H_1)$ is of positive type and moreover admits a valid fill-in (Ω, g_1) whose scalar curvature is strictly positive in a neighborhood U of Σ. (For the latter statement, refer to equation (6) and note that $\rho'(0) > 0$.)

Step 2: We define a metric g_2 on Ω as follows, with the goal of making the boundary mean curvature of g_2 equal to H_2. First, consider a neighborhood $\Sigma \times (-t_0, 0] \subset U$ (where $t = 0$ corresponds to Σ). Define for $x \in \Sigma$ and $t \in (-t_0, 0]$:

$$g_2(x, t) = \rho(t)^2 dt^2 + (1 + tH_2(x))\gamma(x),$$

where $\rho(t)$ is a function satisfying $\rho(0) = 1$ and will be specified later. It is readily checked that g_2 induces on Σ the metric γ and mean curvature H_2. Shrinking t_0 if necessary and choosing $\rho(t)$ bounded below by a positive constant with $\rho'(t) > 0$ sufficiently large, we may arrange g_2 to have strictly positive scalar curvature on $\Sigma \times (-t_0, 0]$. This is readily checked using equation (5). Now, extend g_2 arbitrarily to a smooth metric on Ω (not necessarily preserving nonnegative scalar curvature). Replace U with the smaller neighborhood $\Sigma \times (-t_0, 0]$

To summarize, we have two metrics g_1 and g_2 on the compact manifold Ω, inducing boundary data $(\Sigma, \gamma, \alpha H_1)$ and (Σ, γ, H_2), respectively, each with positive scalar curvature on the neighborhood U of Σ. By compactness, the scalar curvatures of $g_1|_U$ and $g_2|_U$ are bounded below by a constant $R_0 > 0$.

Step 3: Apply Theorem 5 of Brendle–Marques–Neves [9] to produce a metric $\hat{g}$ on Ω satisfying the following properties[2]:
   (i) $R_{\hat{g}}(x) \geq \min\{R_{g_1}(x), R_{g_2}(x)\} - \frac{R_0}{2}$,
   (ii) $\hat{g}$ agrees with g_1 outside of U,
   (iii) $\hat{g}$ agrees with g_2 in some neighborhood of Σ.

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[2]Due to the local nature of the construction, it is clear that we can ignore any connected components of $\partial\Omega$ that are not Σ.

(To apply the theorem, it is crucial that $\alpha H_1 > H_2$.)

By the third condition, $(\Omega, \hat{g})$ is a fill-in of (Σ, γ, H_2). By the first and second conditions, $\hat{g}$ has nonnegative (but not identically zero) scalar curvature and $\partial\Omega \setminus \Sigma$ (if nonempty) is a minimal surface. In particular, $(\Omega, \hat{g})$ is a valid fill-in with positive scalar curvature at some point. $\qquad\square$

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